Defn: Centre of Mass

If $m_1, m_2, m_3 \ldots\ldots$ are the masses of a system of particles situated respectively at points whose position vectors are $\vec{r}_1, \vec{r}_2, \ldots\ldots$ then the point G whose position vector is

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots}{m_1 + m_2 + \ldots\ldots} = \frac{\Sigma m_i\vec{r}_i}{\Sigma m_i}$$

is called the centre of mass (or) mass centre of the system of particles.

Note: Centre of mass of a system is unique.

If the position vectors of the mass centres of several systems of masses $M_1, M_2 \ldots$ are $\vec{R}_1, \vec{R}_2 \ldots\ldots$ then the position vector of the mass centre of all the systems considered together is

$$\vec{R} = \frac{M_1\vec{R}_1 + M_2\vec{R}_2 + \ldots\ldots}{M_1 + M_2 + \ldots\ldots}$$

because $M_1\vec{R}_1 = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots$ and so on.

Centre of gravity:

The resultant of the forces of earth's gravitation acting on a system of particles act always through the mass centre of the system. As such, the mass centre is also called centre of gravity of the system.

Remarks:

1. The C.G of two particles of masses m_1, m_2 is the point which divides the line joining them in the ratio $m_2 : m_1$.

2. For a distribution of mass symmetrical about a point itself. For example, the mass Centres of a uniform sphere and a uniform thin Circular Lamina are their Centres, the mass Centre of a uniform rod is its midpoint.

3. Suppose that several bodies of masses $M_1, M_2, \ldots$ have their mass Centres on a straight line. Then the mass Centre of all the bodies lies on the Same straight line because, if the position Vectors of the individual mass Centres are $x_1\vec{i}, x_2\vec{i} \ldots$ then the p.v. $\vec{R}$ of the entire system is

$$\vec{R} = \frac{M_1 x_1 \vec{i} + M_2 x_2 \vec{i} + \cdots}{M_1 + M_2 + \cdots} = \frac{M_1 x_1 + M_2 x_2 + \cdots}{M_1 + M_2 + \cdots}\,\vec{i}.$$

4. If $(x_1, y_1, z_1), (x_2, y_2, z_2) \ldots$ are the mass Centres of bodies of masses $m_1, m_2, \ldots$ then the mass Centre $(\bar{x}, \bar{y}, \bar{z})$ of all the bodies Considered together is given by

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots}, \quad \bar{y} = \frac{m_1 y_1 + m_2 y_2 + \cdots}{m_1 + m_2 + \cdots}, \quad \bar{z} = \frac{m_1 z_1 + m_2 z_2 + \cdots}{m_1 + m_2 + \cdots}$$

5. For a Continuous mass distribution of total mass M, the p.v $\vec{R}$ of its mass Centre is

$$\vec{R} = \frac{\int \vec{r}\,dm}{\int dm} = \frac{\int \vec{r}\,dm}{M}.$$

Where $\vec{r}$ is the position vector of an element of mass dm. If $\vec{r}$ is $x\vec{i}+y\vec{j}+z\vec{k}$ and if the position vector of the mass centre is $\bar{x}\vec{i}+\bar{y}\vec{j}+\bar{z}\vec{k}$, then

$$\bar{x} = \frac{\int x\,dm}{\int dm}, \quad \bar{y} = \frac{\int y\,dm}{\int dm}, \quad \bar{z} = \frac{\int z\,dm}{\int dm}$$

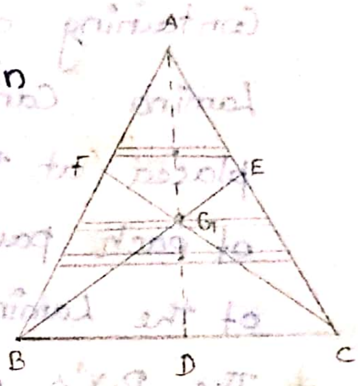
Finding Mass Centre:

Mass Centre can be found by using integration or without using integration.

Finding Mass Centre: [Not Using integration]

Mass Centres of certain simple systems are:

1. Triangular Lamina:



Divide the Lamina ABC into thin strips, all parallel to the side BC. The mass centres of the strips are their midpoints. But these midpoints lie on the median AD. So the mass centre of all the strips also lies on AD. Similarly it lies on the median BE also. Hence the mass centre of the Lamina is the point of intersection of the medians, namely the Centroid of the triangle.

2. Three particles of same mass:

Suppose three particles of masses m, m, m are situated at the points A, B, C whose p.v's are $\bar{a}, \bar{b}, \bar{c}$. By definition, the position vector of their mass centre is $\dfrac{m\bar{a}+m\bar{b}+m\bar{c}}{m+m+m}$ (or) $\dfrac{\bar{a}+\bar{b}+\bar{c}}{3}$. But this

Vector is the position Vector (4) of the Centroid (8)
of the triangle ABC, So the mass Centre of three particles of equal mass placed at A, B, C is at the Centroid of the triangle ABC. So the mass Centre of three particles of equal mass placed at A, B, C is at the Centroid of the triangle ABC.

Note: The mass Centre of a

* The mass Centre of a triangular Lamina ABC is also the Centroid of the triangle ABC.
* Hence, in finding the mass Centre of a system Containing a triangular Lamina, the triangular Lamina Can be replaced by three particles placed at the angular points such that the mass of each particle equals one-third of the mass of the Lamina.
* The p.v's of the midpoints of the sides of the triangle are $\frac{1}{2}(\vec{b}+\vec{c})$, $\frac{1}{2}(\vec{c}+\vec{a})$, $\frac{1}{2}(\vec{a}+\vec{b})$ and the p.v of the mass Centre is

$$\frac{\frac{1}{2}m(\vec{b}+\vec{c})+\frac{1}{2}m(\vec{c}+\vec{a})+\frac{1}{2}m(\vec{a}+\vec{b})}{m+m+m} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}.$$

* Hence the mass Centre of three particle of equal mass situated at the mid-points of the sides of a triangle also is at the Centroid of the triangle.

3. Three particles of certain masses: ⑤ ⑦

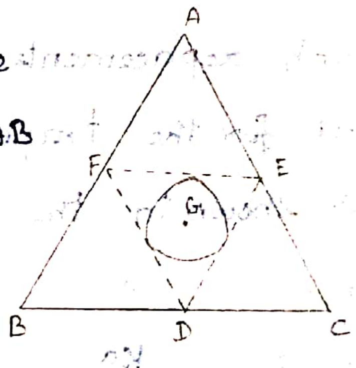
Suppose the positions of three given particles are A, B, C and their masses are proportional to the lengths $AB=c$, $BC=a$, $CA=b$. Then the masses are ka, kb, kc. Now the P.V of the mass centre of the particles is

$$\frac{(ka)\vec{OA}+(kb)\vec{OB}+(kc)\vec{OC}}{ka+kb+kc} \quad (\text{or}) \quad \frac{a\vec{OA}+b\vec{OB}+c\vec{OC}}{a+b+c}.$$

We know that this vector is position vector of the incentre of the triangle ABC. So the mass centre of the three particles is the incentre of the triangle ABC

4. Three uniform rods forming a triangle:

Let the rods be BC, CA, AB and their midpoints be D, E, F Let the masses of the rods be λBC, λCA, λAB where λ is the mass per unit length. Then the P.V of their mass centre is



$$\frac{(\lambda BC)\vec{OD}+(\lambda CA)\vec{OE}+(\lambda AB)\vec{OF}}{\lambda BC+\lambda CA+\lambda AB}$$

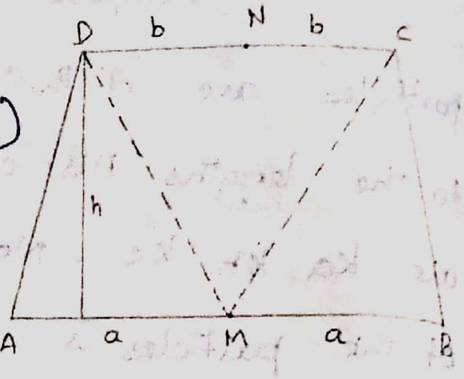
Cancelling λ's and dividing both the numerator and denominator by 2, weget

$$\frac{EF(\vec{OD})+FD(\vec{OE})+DE(\vec{OF})}{EF+FD+DE}$$

Since $EF=\frac{1}{2}BC$, and so on. This shows that the mass centre of the triangular frame is the incentre of the triangle DEF.

5. Lamina in the form of a trapezium:

Let us have the following assumptions:



AB, CD: parallel sides ($AB = 2a$, $CD = 2b$)

h : distance between AB & DC

M, N : Midpoints of AB & DC

σ : Mass per unit area.

Divide the ~~trapbu~~ trapezium into the triangles AMD, DMC, CMB.

Their masses are $(\frac{1}{2}ah)\sigma$, $(\frac{1}{2}\cdot 2b\cdot h)\sigma$, $(\frac{1}{2}ah)\sigma$

$\Rightarrow 3ka,\ 6kb,\ 3ka$ where $k = \frac{1}{6}h\sigma$.

W.K.T a triangular Lamina Can be replaced with three equal particles, each of mass equal to one-third of the mass of the triangle, placed at the angular points. Such replacements for the triangles, AMD, DMC, CMB and for the trapezium Can be Carried out successively as shown in the table.

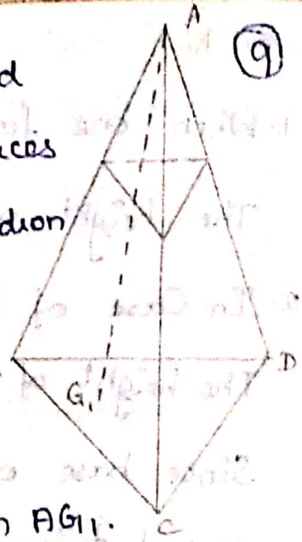
Fig	A	M	B	D	N	C
AMD	ka	ka	–	ka	–	–
MBC	–	ka	ka	–	–	ka
DMC	–	$2kb$	–	$2kb$	–	$2kb$
ABCD	ka	$2k(a+b)$	ka	$k(a+2b)$	–	$k(a+2b)$
ABCD	–	$2k(2a+b)$	–	–	$2k(a+2b)$	

Thus we see that the mass Centre of the trapezium divides MN in the ratio

$2k(a+2b) : 2k(2a+b)$ or $a+2b : 2a+b$.

6. Solid tetrahedron:

Let ABCD be the tetrahedron and G_1, G_2, G_3, G_4 be the centroids of the faces BCD, ACD, ABD, ABC. Divide the tetrahedron into thin triangular lamina, all parallel to the face BCD.



Then due to symmetric, their centroid, that is their mass centres lie on AG_1.

IIIly, it lies on BG_2, CG_3, DG_4.

$\Rightarrow AG_1, BG_2, CG_3, DG_4$ are concurrent.

$\Rightarrow$ The mass centre is the point of concurrence.

Let Q_1, Q_2, Q_3, Q_4 be the points which divide AG_1, BG_2, CG_3, DG_4 in the same ratio 3:1.

Suppose the position vectors of A,B,C,D are $\vec{a}, \vec{b}, \vec{c}, \vec{d}$

Then the p.v of $G_1 = \dfrac{\vec{b}+\vec{c}+\vec{d}}{3}$

Also the p.v of $Q_1 = \dfrac{3\left[\frac{1}{3}(\vec{b}+\vec{c}+\vec{d})\right]+1(\vec{a})}{3+1}$

$= \dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$

Then Q_1, Q_2, Q_3, Q_4 coincide due to symmetry.

$\Rightarrow$ The mass centre is the point of concurrence.

$\therefore$ We have the point which divides the line joining one vertex and the centroid of the opposite face in the ratio 3:1 is the mass centre of the tetrahedron.

Remarks:

1. When one face of the tetrahedron is horizontal,
The height of the mass Centre $= \frac{1}{4}$ [height of the opposite Vertex]

2. In Case of pyramid,
The height of the mass Centre $= \frac{1}{4}$ [height of the pyramid]
Since base of the pyramid can be divided into several triangles.
$\Rightarrow$ The pyramid can be divided into tetrahedrons with the same height of the pyramid.

3. Considering right circular cone to be a limiting Case of a pyramid, the mass Centre of a cone divides the axis in the ratio 3:1.

Problems:

1. D, E, F are the midpoints of the sides BC, CA, AB of a ΔABC. Masses m_1, m_2, m_3 are placed at A, B, C and masses M_1, M_2, M_3 are placed at D, E, F. If the two systems have the same mass Centre, Show that

$$\frac{m_1}{M_2+M_3} = \frac{m_2}{M_3+M_1} = \frac{m_3}{M_1+M_2}.$$

Soln:

Let $\vec{a}, \vec{b}, \vec{c}$ be the p.v's of A, B, C. Then the p.v's of D, E, F are $\frac{\vec{b}+\vec{c}}{2}, \frac{\vec{c}+\vec{a}}{2}, \frac{\vec{a}+\vec{b}}{2}$.

Since the mass Centres of the two systems Coincide,

$$\frac{m_1\vec{a}+m_2\vec{b}+m_3\vec{c}}{m_1+m_2+m_3} = \frac{M_1\cdot\frac{1}{2}(\vec{b}+\vec{c})+M_2\cdot\frac{1}{2}(\vec{c}+\vec{a})+M_3\cdot\frac{1}{2}(\vec{a}+\vec{b})}{M_1+M_2+M_3}$$

$\therefore$ The Coefficients of $\vec{a}, \vec{b}, \vec{c}$ from the left and right sides are equal. Equating the Coefficients of $\vec{a}$. (11)

$$\frac{m_1}{m_1+m_2+m_3} = \frac{M_2+M_3}{2(M_1+M_2+M_3)}$$

$$\Rightarrow \frac{m_1}{M_2+M_3} = \frac{m_1+m_2+m_3}{2(M_1+M_2+M_3)}, \text{ a Constant}$$

$$\therefore \frac{m_1}{M_2+M_3} = \frac{m_2}{M_3+M_1} = \frac{m_3}{M_1+M_2}.$$

2. Three uniform rods form a triangle ABC. If the mass Centre of three particles of masses P, Q, R situated at A, B, C Coincides with the mass Centre of the rods, Show that $\frac{P}{b+c} = \frac{Q}{c+a} = \frac{R}{a+b}$.

Soln:

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$

$$\text{Mass Centre of ABC} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \dots + m_n\vec{r_n}}{m_1+m_2+\dots+m_n}$$

$$= \frac{P\vec{a} + Q\vec{b} + R\vec{c}}{P+Q+R} \rightarrow (1)$$

$$\text{Mass Centre of rods} = \frac{\lambda a\left(\frac{\vec{b}+\vec{c}}{2}\right) + \lambda b\left(\frac{\vec{a}+\vec{c}}{2}\right) + \lambda c\left(\frac{\vec{a}+\vec{b}}{2}\right)}{\lambda a + \lambda b + \lambda c}$$

$$= \frac{a(\vec{b}+\vec{c}) + b(\vec{a}+\vec{c}) + c(\vec{a}+\vec{b})}{2(a+b+c)} \rightarrow (2)$$

given (1) = (2)

Comparing Coefficient of $\vec{a}, \vec{b}, \vec{c}$

$$\frac{P}{P+Q+R} = \frac{b+c}{2(a+b+c)}$$

$$\frac{P}{b+c} = \frac{P+Q+R}{2(a+b+c)}$$

$|||^{ly}$

$$\frac{Q}{a+c} = \frac{P+Q+R}{2(a+b+c)}$$

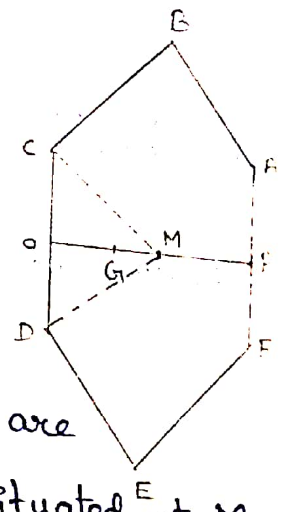
$$\frac{R}{a+b} = \frac{P+Q+R}{2(a+b+c)}$$

$$\Rightarrow \frac{P}{b+c} = \frac{Q}{a+c} = \frac{R}{a+b}$$

3. A rod of length 5a is bent so as to form five sides of a regular hexagon. Show that its Centre of mass is at a distance $a\sqrt{1 \cdot 33}$ from either end of the rod.

Soln:

Let ABCDEF be the hexagon formed.



M → the Centre

A, F → the free ends of the rod

O, P → the midpoint of CD, AF

m → the mass of each sides

The masses m, m of the rods AB, DE are equivalent to a particle of mass 2m situated at M.

$|||^{ly}$ the masses m, m of the rods BC, EF are equivalent to a particle of mass 2m situated at M.

So the system is equivalent to a mass 4m at and a mass m at O. So if G lies on OM and divides it in the ratio 4 : 1.

So, $GM = \frac{OM}{5} = \frac{1}{5}\cdot\frac{\sqrt{3}a}{2}$ ($\because \Delta CDM$ is equilateral)

$$\therefore GA^2 = (GM+MP)^2 + PA^2$$

$$= \left(\frac{\sqrt{3}a}{10} + \frac{\sqrt{3}a}{2}\right)^2 + \frac{a^2}{4}$$

$$= 3a^2\left(\frac{36}{100}\right) + \frac{a^2}{4}$$

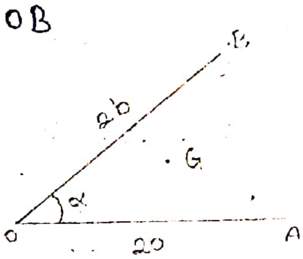
$$= \frac{133}{100}a^2$$

$$GA = \sqrt{1.33}\,a$$

4. OA and OB are two uniform rods of lengths $2a, 2b$. If angle $AOB = \alpha$, Show that the distance of the mass Centre of the rods from O, is $\frac{(a^4 + 2a^2b^2\cos\alpha + b^4)^{1/2}}{a+b}$

Soln:

If $\hat{a}, \hat{b}$ are the unit Vectors along OA, OB then the p.v's of the OA, OB are $a\hat{a}, b\hat{b}$.



If σ is the mass per unit length, then the masses of the rods are $2a\sigma, 2b\sigma$.

So, if G is the mass Centre, then

$$\overline{OG} = \frac{(2a\sigma)a\hat{a} + (2b\sigma)b\hat{b}}{2a\sigma + 2b\sigma}$$

$$= \frac{a^2\hat{a} + b^2\hat{b}}{a+b}$$

$$\therefore OG^2 = \frac{1}{(a+b)^2}(a^2\hat{a} + b^2\hat{b}).(a^2\hat{a} + b^2\hat{b})$$

$$= \frac{a^4 + 2a^2b^2\hat{a}\cdot\hat{b} + b^4}{(a+b)^2}$$

$$OG^2 = \frac{a^4 + 2a^2b^2\cos\alpha + b^4}{(a+b)^2}$$

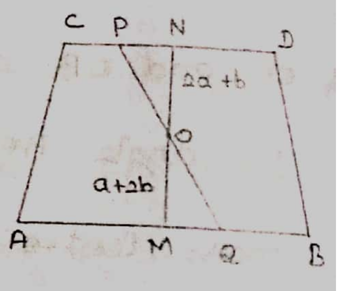
$$OG = \frac{(a^4 + 2a^2b^2\cos\alpha + b^4)^{1/2}}{(a+b)}$$

$\therefore$ Hence the result

5. If one of the parallel sides of a trapezium is double the other, show that the ratio of the distances of the mass centre from the parallel sides is 5 : 4.

Soln:

Let AB & CD be the parallel sides of the trapezium. Let M, N be the midpoints of AB, CD. Let PQ be the perpendicular distance b/w AB & CD. The centre of gravity divides MN in the ratio $a+2b : 2a+b$.



ΔOPN & ΔOQM are similar.

$$\therefore \frac{PO}{OQ} = \frac{ON}{OM} = \frac{2a+b}{a+2b}$$

given, $a = 2b$

$$\therefore \frac{PO}{OQ} = \frac{4b+b}{2b+2b} = \frac{5b}{4b}$$

$PO : OQ = 5 : 4$

6. The form of a lamina is a trapezium ABCD in which AB, CD are parallel and $AB = a$, $CD = b$. Show that, if h is the distance of its mass centre from AB is $\frac{a+2b}{3(a+b)}h$.

Soln:

Let ABCD is the trapezium. Let AB & CD be parallel sides. Let PQ is distance between AB & CD.

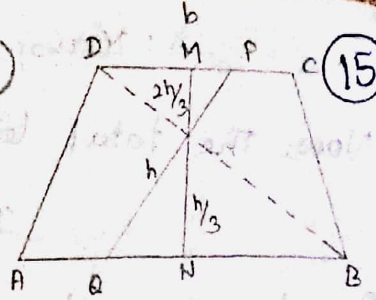
given, $PQ = h$.

Consider Δ^{ls} ABD & BDC

Area : $\frac{1}{2}ah$, $\frac{1}{2}bh$

Their Centroids are at heights $h/3$, $2h/3$

from AB.



AB → x axis

$\bar{y}$ → distance of the Centre of mass of the Lamina

from AB

$$\bar{y} = \frac{h/3 \cdot \frac{1}{2}ah + 2h/3 \cdot \frac{1}{2}bh}{\frac{1}{2}ah + \frac{1}{2}bh}$$

$$\bar{y} = \frac{a+2b}{3(a+b)} \cdot h.$$

Finding mass Centre using integration:

Using the formula for the position Vector of the mass Centre, namely

$$\frac{\int \vec{r}\, dm}{\int dm} = \frac{\int \vec{r}\, dm}{M}$$

The mass Centres of a few homogeneous bodies are:

1. Thin wire in the form of a Circular arc:

Suppose a is the radius of the Circle and 2α is the angle Substended by the wire at the Centre O.

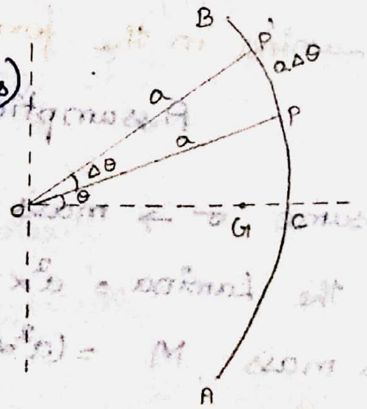
Let us have the following assumptions:

OC : Middle radius (choosen as x axis)

(x,y), (r,θ) : cartesian & polar Coordinates of P, a point on the wire

PP' : Elementary length of the wire

$\Delta\theta$: Angle subtended by PP' at O



λ : Mass per unit length.

Now, the total length of the wire $= a(2\alpha)$

Its mass is $M = a(2\alpha)\lambda$

Due to symmetry of the wire about OC, the mass Centre G of the wire lies on OC

The length of $PP' = a\Delta\theta$

Its mass, $dm = (a\Delta\theta)\lambda$

Its x Coordinate $= a\cos\theta$

Thus we get,

$$OG = \bar{x} = \frac{\int x\,dm}{M} = \frac{\int_{-\alpha}^{\alpha}(a\cos\theta)(a\lambda)\,d\theta}{a(2\alpha)\lambda}$$

$$= \frac{a}{2\alpha}\int_{-\alpha}^{\alpha}\cos\theta\,d\theta$$

$$= \frac{a}{2\alpha}(\sin\theta)_{-\alpha}^{\alpha}$$

$$= \frac{a\sin\alpha}{\alpha}$$

Corollary:

When the wire is Semicircular, $\alpha = \pi/2$

$$OG = \frac{a\sin\pi/2}{\pi/2} = \frac{2a}{\pi}.$$

2. Lamina in the form of a sector of a Circle:

Assumptions are as in above Case. Also assume $\sigma \rightarrow$ mass per unit area. Then the area of the Lamina $= a^2\alpha$

Its mass, $M = (a^2\alpha)\sigma$

The area of the elementary area opp' is

$$\frac{1}{2}\cdot op\cdot op'\sin\Delta\theta \approx \frac{1}{2}a^2\Delta\theta$$

The mass of this area $=\left(\frac{1}{2}a^2\Delta\theta\right)\sigma$

The x- Coordinate of its mass Centre $=\frac{2}{3}a\cos\theta$

So,

$$OG = \frac{\int_{-\alpha}^{\alpha}\left(\frac{2}{3}a\cos\theta\right)\left(\frac{1}{2}a^2\sigma\right)d\theta}{a^2\alpha\sigma}$$

$$= \frac{a}{3\alpha}\int_{-\alpha}^{\alpha}\cos\theta\, d\theta$$

$$= \frac{2a}{3\alpha}\int_{0}^{\alpha}\cos\theta\, d\theta$$

$$= \frac{2a}{3\alpha}\left(\sin\theta\right)_0^{\alpha}$$

$$= \frac{2a\sin\alpha}{3\alpha}$$

Corollary:

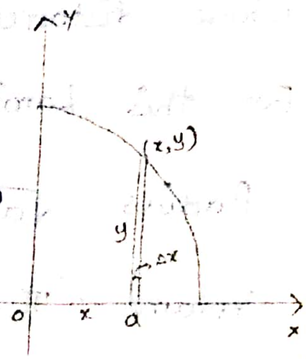
Case(i): In the Case of a quadrant of a Circle, $\alpha = \pi/4$.

Distance of the mass Centre from the Centre

$$= \frac{2a\sin\pi/4}{3(\pi/4)} = \frac{8a\,(1/\sqrt{2})}{3\pi} = \frac{4\sqrt{2}}{3\pi}a$$

Case(ii): For a semicircular Lamina, the distance of the mass Centre from the Centre $= \frac{2a\sin\pi/2}{3(\pi/2)} = \frac{4a}{3\pi}$.

3. Lamina in the form of a quadrant of an ellipse of axes $2a, 2b$:



Corresponding to the elementary strip area $y\Delta x$ as shown in the figure, to use the formula $\bar{x} = \frac{\int x\,dm}{M}$, we have

$$dm = (y\Delta x)\sigma \quad \text{and} \quad M = \left(\frac{1}{4}\pi ab\right)\sigma$$

where σ is the mass per unit area.

So $$\bar{x} = \frac{\int_0^a x(y\sigma\, dx)}{\frac{1}{4}\pi ab\sigma} = \frac{4}{\pi ab}\int_0^a xy\,dx$$

But $x = a\cos\theta$, $y = b\sin\theta$

$dx = -a\sin\theta\, d\theta$

when $x = 0$, $\theta = \pi/2$

when $x = a$, $\theta = 0$

Thus $$\bar{x} = \frac{4}{\pi ab}\int_{\pi/2}^{0}(a\cos\theta)(b\sin\theta)(-a\sin\theta)\,d\theta$$

$$= \frac{4a}{\pi}\int_0^{\pi/2}\sin^2\theta\cos\theta\,d\theta$$

$$= \frac{4a}{\pi}\cdot\frac{1}{3} = \frac{4a}{3\pi}$$

$$\bar{x} = \frac{4a}{3\pi}$$

$|||^{ly}$

$$\bar{y} = \frac{4b}{3\pi}$$

4. Solid hemisphere of radius a:

Divide the hemisphere into thin circular lamina perpendicular to the middle radius OA, where O is the centre of the sphere. Consider the lamina whose distance from O is x and thickness is Δx. For this lamina we have the following:

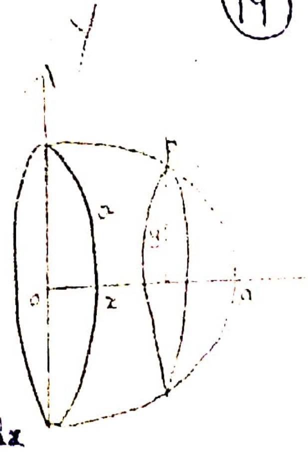
Radius : $\sqrt{a^2 - x^2}$

Volume : $\pi(a^2 - x^2)\Delta x$

Mass : $dm = \pi(a^2 - x^2)\Delta x \rho$

where ρ is mass per unit volume.

Mass Centre : point on OA at a distance x from O.



Thus, if the mass Centre of the hemisphere is G, then

$$OG = \frac{\int x\,dm}{M} = \frac{\int_0^a x\,[\pi(a^2-x^2)\rho]\,dx}{\frac{1}{2}\left(\frac{4}{3}\pi a^3\right)\rho}$$

$$= \frac{3}{2a^3}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{3}{2a^3}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{3}{2a^3}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{3}{2a^3}\left(\frac{a^4}{4}\right)$$

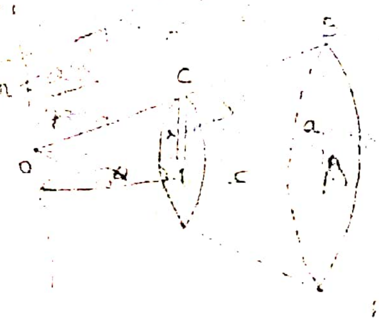
$$OG = \frac{3a}{8}$$

5. Solid right circular Cone of height h:

Divide the Cone into thin circular Laminae perpendicular to its axis OA. Consider the Lamina whose distance from the Vertex O is x.

Thickness $= \Delta x$

For this Lamina, we have the following:



Radius : $\frac{ax}{h}$

Volume : $\pi\left(\frac{ax}{h}\right)^2 \Delta x$

Mass : $dm = \pi\left(\frac{ax}{h}\right)^2 \Delta x \rho = \frac{\rho\pi a^2 x^2}{h^2}\Delta x$

where ρ is the mass per unit Volume

Mass Centre: point on the axis at a distance x from O

Thus, if the mass Centre of the Cone is G, then

$$OG = \frac{\int x\,dm}{M}$$

$$= \frac{\int_0^h \frac{x \rho \pi a^2 x^2}{h^2}\,dx}{\left(\frac{1}{3}\pi a^2 h\right)\rho}$$

$$= \frac{3}{h^3}\int_0^h x^3\,dx$$

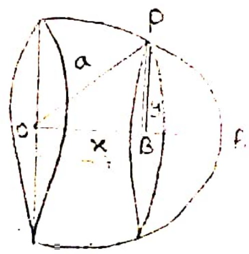
$$= \frac{3}{h^3}\left(\frac{x^4}{4}\right)_0^h$$

$$= \frac{3}{h^3}\left(\frac{h^4}{4}\right)$$

$$OG = \frac{3h}{4}.$$

6. Hemispherical Shell:

Divide the sphere into thin rings whose planes are perpendicular to the middle radius OA. Consider the ring whose Centre is at a distance x from O and thickness is Δs. For this ring we have the following.



Radius : $y = \sqrt{a^2 - x^2}$

Surface area: $(2\pi y)\,\Delta s = (2\pi y)\sqrt{1+y'^2}\,\Delta x$

$$= 2\pi y\sqrt{1+\left(-\tfrac{x}{y}\right)^2}\,\Delta x$$

$$= 2\pi y\sqrt{\frac{y^2+x^2}{y^2}}\,\Delta x$$

$$= 2\pi y\sqrt{\frac{a^2}{y^2}}\,\Delta x = 2\pi a\,\Delta x$$

Mass : $dm = (2\pi a \Delta x)\sigma$, σ being mass per unit area. (21)

Mass Centre : point on OA at a distance x from O.

Thus, if G is the mass Centre of the hemispherical shell,

then,

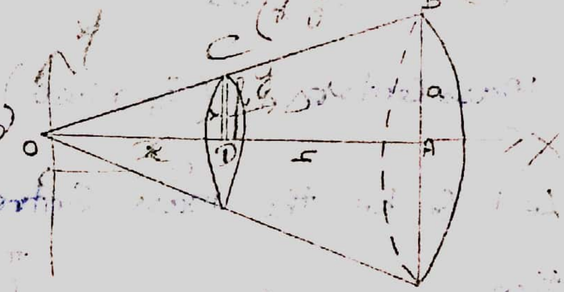
$$OG = \frac{\int x\,dm}{M} = \frac{\int_0^a x(2\pi a\sigma)\,dx}{\frac{1}{2}(4\pi a^2\sigma)} = \frac{1}{a}\int_0^a x\,dm = \frac{1}{2}a.$$

7.) Hollow right Circular Cone of height h :

Now, we Consider a hollow Cone without the base, whose slant side and base radius are l and a.

Divide the Cone into thin Circular ring

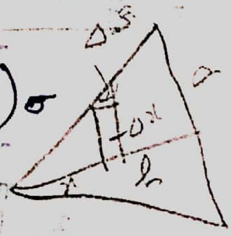
For this Ring we have the following assumptions:



Radius : $y = \frac{ax}{h}$

Surface area : $2\pi \frac{ax}{h} \Delta S$

$$= \frac{2\pi a x}{h} \cdot \frac{l}{h} \Delta x$$



Mass : $dm = \left(\frac{2\pi a}{h^2} l x \Delta x\right)\sigma$

$$OG = \frac{\int x\,dm}{M}$$

$$= \frac{\int_0^h x \frac{2\pi a}{h^2} l x\,dx\,\sigma}{\pi a l \sigma}$$

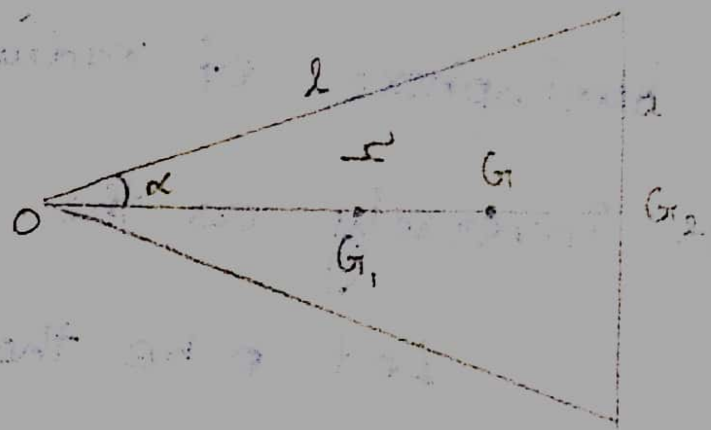
$$= \frac{2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{2}{h^2}\left(\frac{x^3}{3}\right)_0^h = \frac{2}{h^2}\left(\frac{h^3}{3}\right)$$

$$OG = \frac{2}{3}h$$

Problems:

1. S.T the Vertical angle α of a Cone which is such that the C.G of its whole surface area including the base Coincides with the C.G of its Volume is $\sin^{-1} 1/3$.



Soln:

For the Cone, let

a : Base radius

h : Height

l : Slant side

O : Vertex

G_1 : C.G of Curved Surface

G_2 : C.G of base

If G is the C.G of the solid Cone, then

$OG = \frac{3h}{4}$, $OG_1 = \frac{2h}{3}$

$= OG - OG_1$

$GG_2 = \frac{h}{4}$, $GG_1 = \frac{3h}{4} - \frac{2h}{3} = \frac{h}{12}$.

If σ is the mass per unit area, then

Mass at $G_1 = \pi a l \sigma$

Mass at $G_2 = \pi a^2 \sigma$

Taking moments about G,

$$GG_1 (\pi a l \sigma) = GG_2 (\pi a^2 \sigma)$$

$$GG_1 \cdot l = GG_2 \cdot a$$

$$\frac{h}{12} \cdot l = \frac{h}{4} a$$

$$\frac{a}{l} = \frac{1}{3}$$

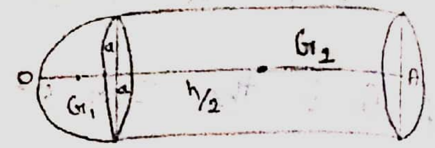
$$\sin\alpha = 1/3$$

$$\alpha = \sin^{-1} 1/3 .$$

2. A solid Circular cylinder is attached to a solid hemisphere of equal base, with bases together. S.T, if the Centre of mass of the solid lies in the Common base, the ratio of the height of the cylinder to the radius of the base is $1:\sqrt{2}$.

Soln:

Let,



h: height of cyclinder

a : radius of cyclinder

OA : Common axis of the solid

$\vec{i}$: Unit Vector along OA

G_1 : Centre of mass of hemisphere

G_2 : Centre of mass of cyclinder

G : Centre of mass of whole solid

σ : mass per unit length

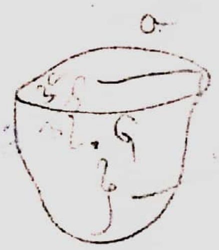
Mass of hemisphere $= \frac{1}{2}\left(\frac{4}{3}\pi a^3\right)\sigma$

Mass of Cylinder $= (\pi a^2 h)\sigma$

Now $\bar{OG} = \dfrac{\left(\frac{2}{3}\pi a^3 \sigma\right)\bar{OG_1} + (\pi a^2 h\sigma)\bar{OG_2}}{\frac{2}{3}\pi a^3\sigma + \pi a^2 h\sigma}$

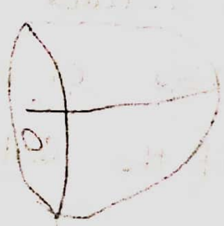
$$a\vec{i} = \frac{\left[\frac{2}{3}a\left(\frac{5a}{8}\right) + h\left(a + h/2\right)\right]\vec{i}}{\frac{2}{3}a + h}$$

$$a = \left(\frac{\frac{5a^2}{12} + ah + \frac{h^2}{2}}{2a + 3h}\right) \times 3$$



$$2a^2 + 3ha = \frac{5a^2}{4} + 3ah + \frac{3h^2}{2}$$

$$\frac{3a^2}{4} = \frac{3h^2}{2}$$



$$h^2 = \frac{a^2}{2}$$

$$h^2/a^2 = 1/2$$

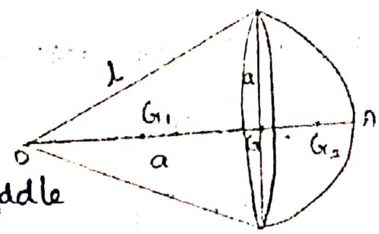
$$h/a = 1/\sqrt{2}$$

$$h : a = 1 : \sqrt{2}$$

3. A solid Cone is attached to a solid hemisphere of equal base, with bases together. If the height of the Cone equals its base radius a, show that the mass Centre of the Combined body is at a distance $7a/6$ from the vertex of the Cone. (26)

Soln:

Let,



O : Vertex of the Cone

OA : Axis of the Cone through middle radius

given, Height of the Cone = base radius ie) h=a

l : slant height of Cone

a : base radius

G_1 : C.G of solid Cone

G_2 : C.G of solid hemisphere

G : C.G of total solid

σ : mass per unit area.

$OG_1 = \frac{3}{4}a$

hemis $= \frac{3a}{8}$

Cone $= \frac{3h}{4} = \frac{3a}{4}$

$(\because h = r = a)$

Mass of Solid cone $= \frac{1}{3}\pi a^3 \sigma$

Mass of solid hemisphere $= \frac{1}{2}\left(\frac{4}{3}\pi a^3\right)\sigma = \frac{2}{3}\pi a^3 \sigma$

Then $\overline{OG} = \dfrac{\frac{1}{3}\pi a^3 \sigma\ \overline{OG_1} + \frac{2}{3}\pi a^3 \sigma\ \overline{OG_2}}{\frac{1}{3}\pi a^3 \sigma + \frac{2}{3}\pi a^3 \sigma}$

$\overrightarrow{OG}\ \vec{i} = \dfrac{OG_1\ \vec{i} + 2 \cdot OG_2\ \vec{i}}{1+2}$

$OG\vec{i} = \dfrac{\left(\frac{3}{4}a + 2 \cdot \frac{11}{8}a\right)\vec{i}}{3}$

$OG = \dfrac{\frac{3a}{4} + \frac{11a}{4}}{3} = \dfrac{14a}{12}$

$OG = \dfrac{7a}{6}$

$OG_2 = OG + GG_2$

$= a + \frac{3a}{8}$

$= \frac{11}{8}a$ //.

$\therefore$ C.G of the whole solid is at a distance $\frac{7a}{6}$ from the Vertex of the Cone.